

# A note on spin chain/string duality

H. Dimov<sup>‡</sup>, R.C. Rashkov<sup>†1</sup>

<sup>†</sup>Department of Physics, Sofia University, 1164 Sofia, Bulgaria

<sup>‡</sup> Department of Mathematics, University of Chemical Technology and Metallurgy, 1756 Sofia, Bulgaria

## Abstract

Recently a significant progress in matching the anomalous dimensions of certain class of operators in N=4 SYM and rotating strings was made. The correspondence was established mainly by using of Bethe ansatz technique applied to the spin  $s$  Heisenberg chain model. In a recent paper Kruczenski (hep-th/0311203) suggested to solve the Heisenberg model by using of sigma model approach. In this paper we generalize the solutions obtained by Kruczenski and comment on the dual string theory. It turns out that our solution is related to a generalized ansatz for rotating strings which can be reduced to the so called Neumann-Rosochatius integrable system. We comment on the spin chain sigma model and string solutions and the possibilities for a more precise formulation of on string/gauge theory correspondence.

## 1 Introduction

Recently an important proposal by Berenstein, Maldacena and Nastase [1] made the first step beyond the supergravity approximation in AdS/CFT correspondence. The same authors showed in [1] how certain operators in SYM theory are directly related to string theory in pp-wave limit. It was suggested that a certain class of "nearly" chiral operators are described by string theory

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<sup>1</sup>e-mail: rash@phys.uni-sofia.bg

in pp-wave background and vice versa. Shortly after this results was given, a proposal for more general treatment was made. Namely, since the BMN proposal concerns states with large energy and R-charge, it was realized that in this particular limit one can find reliable results using semiclassical approximation of string theory around solitonic string configurations executing highly symmetric motion in the target space-time. In this case the corresponding gauge theory operators are not supposed to be nearly chiral but they are generically non-chiral [2]. This make the method of rotating strings even more involving. While on string theory side many rotating strings solutions in  $AdS_5 \times S^5$  and other [3], less supersymmetric backgrounds were quickly found [7], the relation to the gauge theory is still not given in such a direct way as in the case of string theory in pp-wave background. Nevertheless, some important steps were made. It was observed by Minahan and Zarembo [9] that SYM theory can be mapped to spin chain integrable system. This turns out to be great advantage because following [12], one can diagonalize the matrix of the anomalous dimensions by using the algebraic Bethe ansatz technique. More specifically, let us consider operators in  $\mathcal{N} = 4$  SYM theory. The R-symmetry group is  $SO(6)$  whose bosonic sector contains six real scalar fields in the adjoint,  $\phi^A$ ,  $A = 1, \dots, 6$ , which can be organize in three complex scalar fields,  $X = \phi^1 + i\phi^2$ ,  $Y = \phi^3 + i\phi^4$ ,  $Z = \phi^5 + i\phi^6$ . The operators in question are of the form  $tr(X^{J_1}Y^{J_2}Z^{J_3})$  where  $J_i$  are large. For instance, significant progress in pp-wave string/SYM correspondence was made in the sector of operators with large number of  $Z$ 's and two other scalars, considered as impurities<sup>2</sup>,

$$\mathcal{O} \sim tr(XYZ \dots Z). \quad (1.1)$$

We note that in rotating strings approach [2, 4, 9] the form of the operators (1.1) is more general as mentioned above, namely

$$\mathcal{O} \sim tr(XZ \dots YY \dots Z \dots Z) \quad (1.2)$$

In both approaches in order to obtain the anomalous dimensions one must diagonalize the action of the dilatation operator acting on these operators. One loop dilatation operator, for instance, has the form [14, 15]

$$D_2 = l \sum_{i=1}^J \left( \frac{1}{4} - \vec{S}_i \vec{S}_{i+1} \right), \quad (1.3)$$

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<sup>2</sup>for three impurities see also [13]

i.e., it acts as permutation operator between near neighborhood operators. One can then map the operators of the form (1.2) into Heisenberg spin chain system which is known to be integrable. For operators of two species,  $X$  and  $Z$  for instance there exists the identification

$$ZZX \dots ZXX \Leftrightarrow |\uparrow\uparrow\downarrow \dots \uparrow\downarrow\downarrow\rangle, \quad (1.4)$$

i.e., we identify the  $Z$  fields with spin up and  $X$  fields with spin down. A powerful tool in studying this integrable system is Bethe ansatz technique which was used to this particular problem first in [9]. It was observed later [11] that the anomalous dimension computed in spin chain model in the first loop approximation and those obtained from rotating strings agrees exactly. This suggests that one can map (non-locally perhaps) the integrable spin chain model to the string sigma model [5]. An important step toward the clarification of the hypothesis that the continuous spin chain action can be mapped directly to the string sigma model action was recently made by Kruczenski [16]. The starting point is the Heisenberg spin chain in its ferromagnetic phase. Taking the length of the spin chain  $J$  to be very large one obtains continuous ferromagnetic sigma model. It was argued that the resulting action coincides with rotating string sigma model in certain limit and a consistency check for a simple solution to the spin chain and rotating string sigma model was given.

In this short note we consider more general solution to the spin chain sigma model. In our case it turns out that the corresponding rotating string ansatz is not that of Neumann integrable system as in [16]<sup>3</sup> but of so called Neumann-Rosochatius integrable system [18]. We analyze the solutions to these systems and find out that, up to one loop, they agree exactly. Although one can expect that there should be some rotating string configuration that corresponds to the more general solutions of the spin chain model, it seems that our result gives a non-trivial check of the conjecture made in [16]. It shows also that the map between the two models is not that straightforward as it might seem at first glance.

In the next section we give a brief review of the large spin chain length limit which corresponds to the continuous limit of Heisenberg spin chain sigma model and the particular limit of the Polyakov string sigma model as proposed in [16]. In next section we find a more general solution to the spin chain model. The corresponding rotating string ansatz is found and

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<sup>3</sup>For more details see also [17]

discussed. In the concluding section we comment on our results and some open problems.

## 2 Review of spin chain/strings duality

In this section we give a brief review of the continuous spin chain sigma model and its relation to string sigma model. It is well known that the Heisenberg model is related to the gauge theories in the large  $N$  limit. By making use of holographic correspondence, the relation between spin chains and string theory was recently studied [15, 9]. These authors successfully used the algebraic Bethe ansatz (see for review [12]) to obtain perfect matching of the anomalous dimensions (in one loop) obtained from string theory (the calculations are actually valid to all loops) and those obtained from gauge theory side. In the same time, however, there are string solutions with very complicated relations between the energy and spins (see for instance [19]) which make the calculation of the anomalous dimensions very hard. To obtain the corresponding gauge theory counterpart, it is desirable to have more direct way to establish the correspondence between the two theories.

Recently Kruczenski suggested such a correspondence based on the equivalence of the continuous ferromagnetic sigma model (which is equivalent to spin  $s$  Heisenberg spin chain) and string sigma model in specific limit [16]. In this section we will follow that line of presentation.

We start with the realization of the spin chain sigma model. The easiest way is may be to use the coherent states on the target space  $SU(2)/U(1) = S^2$  [20, 21], i.e. by using  $\vec{n}$  field formalism

$$|\vec{n}\rangle = e^{iS_z\phi} e^{iS_y\theta} |ss\rangle. \quad (2.1)$$

We will use the standard parametrization of  $S^2$

$$\vec{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \quad (2.2)$$

$$S_z|ss\rangle = s|ss\rangle, \quad (2.3)$$

and

$$\vec{n}^2 = 1. \quad (2.4)$$

Starting from the standard lagrangian

$$\mathcal{L} = \frac{s}{4} (\partial_\mu \vec{n})^2 \quad (2.5)$$

and introducing canonical conjugate variable

$$\vec{p} = \frac{s}{2} \partial_t \vec{n} \times \vec{n} \quad (2.6)$$

one can easily deduce the canonical Poisson brackets

$$\{p^a, p^b\} = \varepsilon^{abc} p^c \delta(x - x')$$

$$\{p^a, n^b\} = \varepsilon^{abc} n^c \delta(x - x') \quad (2.7)$$

$$\{n^a, n^b\} = 0. \quad (2.8)$$

To make contact with the spin chain, we regularize the model as follows

$$\vec{p}_k = \Delta \vec{p}(\sigma); \quad \vec{n}_k = \vec{n}(\sigma); \quad \sigma = k\Delta.$$

Then the sigma model variables in large spin limit can be realized by a pair of spin variables  $S_{2k-1}, S_{2k}$  as follows

$$p_k = S_{2k-1} + S_{2k}$$

$$n_k = \frac{1}{2s}(S_{2k} - S_{2k-1}) \quad (2.9)$$

On the other hand, in order to obtain the sigma model effective action one can follow [20] and use path integral. The result is

$$S(\vec{n}) = s \sum_k \int dt \int_0^1 d\tau (\vec{n}_k \cdot \partial_\mu \vec{n}_k \times \partial_\nu \vec{n} \varepsilon^{\mu\nu}) - \frac{\tilde{l}s^2}{2} \int dt \sum_k |\vec{n}_k - \vec{n}_{k+1}|^2 \quad (2.10)$$

We are interested, however, in the limit of large spin  $J$ , which is actually a long wave limit. In this limit the variable  $\sigma$  becomes continuous and is running from 0 to  $J$ . By making use of the explicit parametrization of the  $\vec{n}$  field and the standard formula  $\Delta \sum_k = \int d\sigma$ , one can obtain the following effective action

$$S(\vec{n}) = s \int dt d\sigma \int_0^1 d\tau \sin \theta \partial_\mu \phi \times \partial_\nu \theta \varepsilon^{\mu\nu} - \frac{\tilde{l}s^2}{2} \int d\sigma dt [(\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2]. \quad (2.11)$$

We will not go into details here but only point out that the ground state of the system is ferromagnetic and only long range fluctuations take place.

We are actually interested in the conserved quantities, energy and spin, which define the anomalous dimensions in the gauge theory. These are given by

$$S_z = P_\phi = -s \int d\sigma \int d\tau \sin \theta \partial_\tau \theta = -s \int d\sigma \cos \theta \quad (2.12)$$

$$J = \int_0^J d\sigma \quad (2.13)$$

$$H = \frac{\tilde{l}s^2}{2} \int d\sigma [(\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2]. \quad (2.14)$$

To bring the action into a form adapted to our analysis we use that the volume two-form  $\omega_2$  is exact and integrate the Wess-Zumino term by part (excluding the poles of the sphere where the usual singularity appears; since our analysis is at classical level it is insignificant) to obtain

$$S(\vec{n}) = -s \int d\sigma dt \cos \theta \partial_t \phi - \frac{\tilde{l}s^2}{2} \int d\sigma dt [(\partial_\sigma \theta)^2 + \sin^2 \theta (\partial_\sigma \phi)^2]. \quad (2.15)$$

From this action one can read off the equations of motion

$$\sin \theta \partial_t \theta + \tilde{l}s \partial_\sigma (\sin^2 \theta \partial_\sigma \phi) = 0 \quad (2.16)$$

$$\sin \theta \partial_t \phi + \tilde{l}s \partial_\sigma^2 \theta - \tilde{l}s \sin \theta \cos \theta (\partial_\sigma \phi)^2 = 0 \quad (2.17)$$

supplemented with the boundary conditions

$$\phi(\sigma = J, t) = \phi(\sigma = 0, t), \quad \theta(\sigma = J, t) = \theta(\sigma = 0, t). \quad (2.18)$$

In [16] the spins and the energy for a particular ansatz were calculated. It was demonstrated that the energy  $E$  and the ratio  $J_2/J$  for this particular solution exactly coincide with those obtained by using of Bethe ansatz results [15]. On the other hand, in [19] it was proven that the Bethe ansatz results and string theory calculations exactly agrees in one loop. To be more specific, the ansatz used in [16] to solve the equations of motion is

$$\partial_\sigma \phi = 0$$

which, by using (2.16), immediately implies that  $\partial_t \theta = 0$  and therefore

$$\partial_t \phi = \omega.$$

With this ansatz the equations of motion (2.16) and (2.17) are greatly simplified and the conserved quantities defining the anomalous dimensions can be easily calculated

$$S_z = -s\sqrt{\frac{2}{b}}\{2E(x) - K(x)\}$$

$$J = \frac{a+b}{2b}, \quad x = \frac{a+b}{2b}$$

and

$$\frac{J_2}{J} = \left(s + \frac{1}{2}\right) - 2s\frac{E(x)}{K(x)}$$

$$E = \frac{\tilde{l}}{J}32s^2K(x)[E(x) - (1-x)K(x)]$$

which is exactly the same as the one loop result from rotating string in [19]<sup>4</sup>.

We will focus now on the string sigma model. An interesting observation is that the spin chain picture looks like rotating string but with no motion of the center of mass (which actually is a feature of pp-wave limit). This is an useful instruction to look for rotating string solution in pp-wave like metric. We are looking for two spin solution in  $S^5$  part of  $AdS_5 \times S^5$  background which means that one should take  $S^3 \subset S^5$ . The relevant part of the metric defining string sigma model is

$$ds^2 = -dt^2 + d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2. \quad (2.19)$$

The string solution was obtained in [19] but in order to compare the result with spin chain, it is useful to make two steps change of variables:

$$\phi_1 = \varphi_1 + \varphi_2, \quad \phi_2 = \varphi_1 - \varphi_2$$

and

$$\varphi \rightarrow t + \varphi_1, \quad (2.20)$$

which yield

$$ds^2 = 2dt(d\varphi_1 + \cos(2\psi)d\varphi_2) + d\psi^2 + d\varphi_1^2 + d\varphi_2^2 + 2\cos(2\psi)d\varphi_1d\varphi_2 \quad (2.21)$$

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<sup>4</sup>For more details see [16].

Taking  $t = \kappa\tau$  the Polyakov action in the above background becomes<sup>5</sup>

$$S = \frac{R^2}{4\pi\alpha'} \int [2\kappa\dot{\varphi}_1 + \dot{\psi}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\cos(2\psi)\kappa\dot{\varphi}_2 + 2\cos(2\psi)\dot{\varphi}_1\dot{\varphi}_2 - \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1'\varphi_2'] \quad (2.22)$$

with the corresponding Virasoro constraints

$$2\kappa\varphi_1' + \dot{\psi}\psi' + \dot{\varphi}_1\varphi_1' + \dot{\varphi}_2\varphi_2' + 2\cos(2\psi)\kappa\varphi_2' + 2\cos(2\psi)\dot{\varphi}_1\varphi_2' + 2\cos(2\psi)\dot{\varphi}_2\varphi_1' = 0, \quad (2.23)$$

$$2\kappa\dot{\varphi}_1 + \dot{\psi}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\cos(2\psi)\kappa\dot{\varphi}_2 + 2\cos(2\psi)\dot{\varphi}_1\dot{\varphi}_2 + 2\cos(2\psi)\varphi_1'\varphi_2' + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 = 0. \quad (2.24)$$

A particular class of two spins solutions corresponding to rotating strings in  $S^3 \subset S^5$  were studied in details in [19]. According to [2] the results are reliable in the limit of large energy and spins, i.e. the limit where the semiclassical approximation is valid. Therefore, we are looking for the terms in the Polyakov action contributing to this particular limit. As it was argued in [16], the case of large spin and energy corresponds to the limit  $\kappa \rightarrow \infty$ . The non-trivial contribution actually comes from the limit

$$\kappa \rightarrow \infty, \quad \dot{X}^\mu \rightarrow 0, \quad \kappa\dot{X}^\mu = \text{fixed} \quad (2.25)$$

The terms in (2.22), (2.23) and (2.24) which survive this limit give the following resulting action and constraints

$$S = \frac{R^2}{4\pi\alpha'} \int [2\kappa\dot{\varphi}_1 + 2\cos(2\psi)\kappa\dot{\varphi}_2 - \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1'\varphi_2'] \quad (2.26)$$

and

$$\kappa\varphi_1' + \cos(2\psi)\kappa\varphi_2' = 0, \quad (2.27)$$

$$2\kappa\dot{\varphi}_1 + 2\cos(2\psi)\kappa\dot{\varphi}_2 + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2\cos(2\psi)\varphi_1'\varphi_2' = 0. \quad (2.28)$$

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<sup>5</sup>We follow the notations of [16]



The field  $\varphi_1$  somehow decouples from the system being completely determined from the constraints (2.27) and (2.28). Finding  $\varphi_1$  from the constraints and substituting it in the action (2.27) we obtain

$$S = \frac{R^2}{4\pi\alpha'} \int [2\kappa\dot{\varphi}_1 + 2\cos(2\psi)\kappa\dot{\varphi}_2 - \psi'^2 - \sin^2(2\psi)\varphi_2'^2] \quad (2.29)$$

If we compare this action to the action for spin chain (2.15) we see that both are completely equivalent. Actually, they coincide if we make use of the identification

$$\tilde{\sigma} = \frac{R^2}{4\pi\alpha'} 2\kappa\sigma, \quad \varphi_2 = -\frac{1}{2}\phi, \quad \psi = \frac{1}{2}\theta \quad (2.30)$$

and the well know relation  $l = R^4/\alpha'^2$ .

The rotating string solution that corresponds to the spin chain solution was investigated by Frolov and Tseytlin [19]. The ansatz used there is

$$t = \kappa\tau, \quad \phi_1 = \omega_1\tau, \quad \phi_2 = \omega_2\tau, \quad \theta = \theta(\sigma) \quad (2.31)$$

with the following expressions for the energy and spins

$$E = \kappa, \quad J_1 = \frac{2\omega_1}{\sqrt{\omega_{12}^2}} E(x) \quad (2.32)$$

where  $x = (\kappa^2 - \omega_1^2)/\omega_{12}^2$  and  $E(x)$  is the complete elliptic integral of second kind. The periodicity condition determines  $\omega_{12}$  in terms of complete elliptic integral of first kind

$$\sqrt{\omega_{12}^2} = \frac{2}{\pi} K(x) \quad (2.33)$$

We conclude this section referring for details to [19, 16].

### 3 A new case of spin chain/string correspondence

In this section we generalize the solution to the spin chain sigma model obtained in [16] and discuss on the corresponding string solutions. As mentioned above, the spin  $s$  Heisenberg model has the following equations of motion

$$\dot{\theta} \sin \theta + \tilde{l}s\partial_\sigma[\sin^2 \theta \partial_\sigma \phi] = 0 \quad (3.1)$$

$$\dot{\phi} \sin \theta + \tilde{l}s\theta'' - \tilde{l}s \sin \theta \cos \theta (\partial_\sigma \phi)^2 = 0 \quad (3.2)$$

We look for more general solution of the equations of motion (3.1) and (3.2) requiring that  $\theta$  is function of  $\sigma$  only, i.e.

$$\dot{\theta} = 0 \quad (3.3)$$

With no other restrictions, one can find that  $\phi$  satisfies the relation

$$\phi' = \frac{A}{\sin^2 \theta}. \quad (3.4)$$

Substituting (3.4) in (3.2) we obtain

$$\dot{\phi} \sin \theta + \tilde{l}_s \theta'' - \tilde{l}_s \frac{A \cos \theta}{\sin^3 \theta} = 0 \quad (3.5)$$

From (3.5) we can find the equation that determines completely  $\phi$

$$\begin{aligned} \ddot{\phi} &= 0, \\ \dot{\phi}' &= 0 \end{aligned} \quad (3.6)$$

with the obvious solution

$$\phi = \omega \tau + \hat{\phi}(\sigma); \quad \hat{\phi}' = \frac{A}{\sin^2 \theta}. \quad (3.7)$$

Using (3.7) we derive the final equation for  $\theta$  variable

$$\omega \sin \theta + \tilde{l}_s \theta'' - \tilde{l}_s \frac{A \cos \theta}{\sin^3 \theta} = 0 \quad (3.8)$$

As a result we have simple equations for  $\phi$  and  $\theta$  that determine the invariants of the model - the energy and spins. Let us concentrate on the solution of (2.9). Multiplying (3.8) by  $\theta'$  and integrating the resulting equation once we get

$$\left( \frac{d\theta}{d\sigma} \right)^2 + \frac{A^2}{\sin^2 \theta} - \frac{2\omega}{\tilde{l}_s} \cos \theta = B. \quad (3.9)$$

If we denote

$$\xi = \cos \theta, \quad (3.10)$$

one can rewrite (3.9) in the form

$$\left( \frac{d\xi}{d\sigma} \right)^2 = \alpha \xi^3 - B \xi^2 - \alpha \xi + B - A^2, \quad (3.11)$$

where we define

$$\alpha = -\frac{2\omega}{\tilde{l}_s} > 0; \quad i.e. \quad \omega < 0. \quad (3.12)$$

It is useful to define a new variable by

$$\xi = \frac{B}{3\alpha} + x. \quad (3.13)$$

In terms of the new variable  $x$  the equation (3.11) takes the form

$$\left(\frac{dx}{d\tilde{\sigma}}\right)^2 = 4x^3 - g_2x - g_3, \quad (3.14)$$

where

$$\tilde{\sigma} = \frac{\sqrt{\alpha}}{2}\sigma, \quad g_2 = 4\frac{B + 3\alpha^2}{3\alpha^2}. \quad (3.15)$$

$$g_3 = \frac{4}{3\alpha} \left( 3A^2 + \frac{2B^3}{9\alpha^2} - 2B \right). \quad (3.16)$$

The equation (3.14) defines an elliptic curve in Weierstrass form. The solution of the equation defines an elliptic integral whose inverse is Weierstrass or Jacobi elliptic functions. Since we are looking for finite periodic solutions, the integration constants should be adjusted so that the solution is Jacobi elliptic function. Let us denote by  $e_1, e_2, e_3$  the roots of the right hand side of (3.14) which, as it is well known from the theory of elliptic functions, satisfy the relations

$$\sum_k e_k = 0, \quad e_1e_2e_3 = \frac{1}{4}g_3, \quad (3.17)$$

$$e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2. \quad (3.18)$$

One can write then (3.14) in the form

$$\left(\frac{dx}{d\tilde{\sigma}}\right)^2 = 4(x - e_1)(x - e_2)(x - e_3). \quad (3.19)$$

It is easy to bring eq.(3.19) into Jacobi form by using the transformation

$$x = e_1 + e_{21}\eta^2. \quad (3.20)$$

Finally, we end up with

$$\left(\frac{d\eta}{d\tilde{\sigma}}\right)^2 = e_{31}(1 - \eta^2)(1 - \kappa\eta^2), \quad (3.21)$$

where the modulus  $\kappa$  is given by  $\kappa = e_{21}/e_{31}$ , and  $e_{nm} = e_n - e_m$ . The finite periodic solution to eq. (3.21) satisfying  $\eta(0) = 0$  is

$$\eta = sn(\sqrt{e_{31}}\tilde{\sigma}, \kappa) = sn\left(\frac{\sqrt{\alpha e_{31}}}{2}\sigma, \kappa\right) \quad (3.22)$$

and the expression for  $\cos\theta$  becomes

$$\cos\theta = \frac{B}{3\alpha} + e_1 + e_{21}sn^2\left(\frac{\sqrt{\alpha e_{31}}}{2}\sigma, \kappa\right) \quad (3.23)$$

In addition we have to satisfy the boundary conditions

$$\theta(\sigma = J, t) = \theta(\sigma = 0, t)$$

which determines  $\alpha$  in terms of the complete elliptic integral of first kind

$$\frac{\sqrt{\alpha e_{31}}}{2}J = 2K(\kappa), \quad \text{or} \quad J = \frac{4K(\kappa)}{\sqrt{\alpha e_{31}}} \quad (3.24)$$

and

$$\cos\theta = \frac{B}{3\alpha} + e_1 + e_{21}sn^2\left(\frac{2K(\kappa)}{J}\sigma, \kappa\right) \quad (3.25)$$

Having the solution (3.25) one can find the conserved quantities as energy and spins. First of all the total spin  $J$  is already (consistently) determined by the periodicity condition

$$J = \frac{4K(\kappa)}{\sqrt{\alpha e_{31}}}. \quad (3.26)$$

The calculation of  $S_z$  is also straightforward

$$\begin{aligned} S_z &= \frac{J_2 - J_1}{2} = -s \int_0^J \cos\theta d\sigma = -s \left[ \frac{B}{3\alpha} + e_1 + e_{31} \left(1 - \frac{E(\kappa)}{K(\kappa)}\right) \right] \\ &= -s \frac{4\varpi}{\sqrt{\alpha}} \left[ \frac{B}{3\alpha} + e_3 - e_{31} \frac{E(\kappa)}{K(\kappa)} \right], \end{aligned} \quad (3.27)$$

where  $\varpi$  is the primitive period of the Weierstrass  $\wp$  function. The energy can be also easily determined as follows. First, we use (3.4) and (3.9) to get

$$\begin{aligned} E &= \frac{\tilde{l}s^2}{2} \int_0^J [(\partial_s \theta)^2 + \sin^2(\partial_s \phi)^2] = \frac{\tilde{l}s^2}{2} \int_0^J [B - \alpha \cos \theta] \\ &= \frac{\tilde{l}s^2}{2} \left[ BJ + \frac{\alpha}{s} S_z \right]. \end{aligned} \quad (3.28)$$

Using (3.26) and (3.27) we find for the energy the next expression

$$E = \frac{2\tilde{l}s^2\varpi}{\alpha} \left[ \frac{2B}{3} - \alpha e_3 + \alpha e_{31} E(\kappa) \right]. \quad (3.29)$$

The spins  $J_i$  are found to be

$$J_1 = \frac{2\varpi}{\sqrt{\alpha}} \left[ 1 + 2s \left( \frac{B}{3\alpha} + e_3 \right) - 2se_{31} \frac{E(\kappa)}{K(\kappa)} \right] \quad (3.30)$$

and

$$J_2 = \frac{2\varpi}{\sqrt{\alpha}} \left[ 1 - 2s \left( \frac{B}{3\alpha} + e_3 \right) + 2se_{31} \frac{E(\kappa)}{K(\kappa)} \right] \quad (3.31)$$

To summarize, we found explicit expressions for the characteristics of the spin chain sigma model that determine the anomalous dimensions.

We turn now to the string side. The consideration of the Polyakov action in the limit  $\hat{\kappa} \rightarrow \infty, \dot{X} \rightarrow 0, \hat{\kappa}\dot{X} = \text{fixed}$  (here we changed the notation  $t = \hat{\kappa}\tau$ ) ended up in the previous section with the expression

$$S = \frac{R^2}{4\pi\alpha} \int \left[ \alpha \hat{\kappa} \dot{\varphi}_1 + 2 \cos(2\psi) \hat{\kappa} \dot{\varphi}_2 - \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2 \cos(2\psi) \varphi_1' \varphi_2' \right] \quad (3.32)$$

with the conformal constraints

$$\begin{aligned} 2\hat{\kappa}\varphi_1' + 2\cos(2\psi)\hat{\kappa}\varphi_2' &= 0, \\ 2\hat{\kappa}\dot{\varphi}_1 + 2\cos(2\psi)\hat{\kappa}\dot{\varphi}_2 + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2\cos(2\psi)\varphi_1'\varphi_2' &= 0. \end{aligned} \quad (3.33)$$

Since we found more general solution for the spin chain allowing  $\phi$  to depend on  $\sigma$ , we expect that the corresponding string solutions are also more general, allowing  $\sigma$  dependence for  $\varphi_i$ . If so, due to (3.33) the variables  $\varphi_i$  are related by

$$\varphi_1' = -\cos(2\psi)\varphi_2'. \quad (3.34)$$

If we substitute (3.34) into (3.32) we will obtain again (2.29) which after identification  $\varphi_2 = -\phi/2, \psi = \theta/2$  and  $\tilde{\sigma} = R^2 2\hat{\kappa}\sigma/4\pi\alpha'$  coincides again with the action for the spin chain model. Assuming that  $\theta$  is  $\tau$  independent, a natural ansatz for  $\varphi_i$  is

$$\begin{aligned}\varphi_1 &= \omega_1\tau + \alpha_1(\sigma) \\ \varphi_2 &= \omega_2\tau + \alpha_2(\sigma).\end{aligned}\tag{3.35}$$

In contrast to the ansatz (2.31) in the previous section, which actually reduces the problem to the Neumann integrable system [17], the ansatz (3.35) is related to the so called Neumann-Rosochatius (N-R) integrable system [18]. Hence, we are to consider two spin string solution of N-R system (we are confined on  $S^3 \subset S^5$  and therefore we can have at most two spins  $J_2$  and  $J_2$ ).

The most general ansatz studied in [18] is

$$\phi_i = \omega_i\tau + \alpha_i(\sigma), \quad i = 1, 2, 3; \quad \psi = \psi(\sigma), \quad \gamma = \gamma(\sigma)\tag{3.36}$$

where the angles (3.36) parameterize  $S^5$  as follows

$$\begin{aligned}X_1 + iX_2 &= \sin \gamma \cos \psi e^{i\phi_1} = z_1(\sigma)e^{\omega_1\tau}, \\ X_3 + iX_4 &= \sin \gamma \sin \psi e^{i\phi_2} = z_2(\sigma)e^{\omega_2\tau}, \\ X_5 + iX_6 &= \cos \gamma e^{i\phi_3} = z_3(\sigma)e^{\omega_3\tau}.\end{aligned}\tag{3.37}$$

It is useful to introduce variables  $r_i(\sigma)$  by

$$z_i(\sigma) = r_i(\sigma)e^{i\alpha_i(\sigma)}, \quad i = 1, 2.\tag{3.38}$$

In our case we have  $\gamma = \pi/2$  and  $\alpha_3 = 0$  which means that

$$\begin{aligned}r_1(\sigma) &= \cos \psi, \\ r_2(\sigma) &= \sin \psi, \quad r_3 = 0.\end{aligned}\tag{3.39}$$

The string lagrangian for the relevant variables is

$$\begin{aligned}\mathbb{L} &= \frac{1}{2} \sum (z'_i z_i^{\star'} - \omega_i^2 z_i z_i^{\star}) + \frac{1}{2} \Lambda (\sum z_i z_i^{\star} - 1) \\ &= \frac{1}{2} \sum (r_i'^2 + r_i^2 \alpha_i'^2 - \omega_i^2 r_i^2) + \frac{1}{2} \Lambda (\sum r_i^2 - 1).\end{aligned}\tag{3.40}$$

The equation of motion for  $\alpha_i$  gives

$$\alpha_i' = \frac{v_i}{r_i^2}, \quad i = 1, 2.\tag{3.41}$$

Substitution of (3.41) into (3.40) leads to the so called Neumann-Rosochatius integrable system defined by the lagrangian

$$\mathbb{L} = \frac{1}{2} \sum \left[ r_i'^2 + \frac{v_i^2}{r_i^2} - \omega_i^2 r_i^2 \right] + \frac{1}{2} \Lambda (\sum r_i^2 - 1). \quad (3.42)$$

The solution of N-R system was studied in [18] by using elliptic coordinates. For our purpose, however, it is more convenient to study the system in global coordinates.

The equations of motion derived from (3.42)

$$r_i'' = -\omega_i^2 r_i - \frac{v_i^2}{r_i^3} - r_i \sum \left( r_j'^2 - \omega_i^2 r_j^2 + \frac{v_j^2}{r_j^2} \right) \quad (3.43)$$

can be written in global coordinates using the relations (3.39). Multiplying the first equation in (3.42) by  $r_2$  and subtracting the second one multiplied by  $r_1$ , one finds

$$\psi'' + \omega_{21}^2 \sin \psi \cos \psi + \frac{v_2^2 \cos \psi}{\sin^3 \psi} - \frac{v_1^2 \sin \psi}{\cos^3 \psi} = 0, \quad (3.44)$$

where

$$\omega_{21}^2 = \omega_2^2 - \omega_1^2. \quad (3.45)$$

One can integrate (3.44) once to obtain

$$\psi'^2 + \omega_{21}^2 \sin^2 \psi - \frac{v_2^2}{\sin^2 \psi} - \frac{v_1^2}{\cos^2 \psi} = \hat{A}, \quad (3.46)$$

or, equivalently

$$\left( \sin \psi \cos \psi \frac{d\psi}{d\sigma} \right)^2 = v_2^2 + (\hat{A} - v_{21}^2) \sin^2 \psi - (\hat{A} + \omega_{21}^2) \sin^4 \psi + \omega_{21}^2 \sin^6 \psi, \quad (3.47)$$

where  $v_{21}^2 = v_2^2 - v_1^2$ . Defining the variable

$$\xi_{str} = \sin^2 \psi \quad (3.48)$$

we find the equation of motion as an elliptic curve equation in Weierstrass form

$$\left( \frac{d\xi_{str}}{d\sigma} \right)^2 = 4[\omega_{21}^2 \xi_{str}^3 - (\omega_{21}^2 + \hat{A}) \xi_{str}^2 - (v_{21}^2 - \hat{A}) \xi_{str} + v_2^2]. \quad (3.49)$$

One can bring the equation (3.49) into Weierstrass form by defining a new variable

$$\xi_{str} = \frac{\omega_{21}^2 + \hat{A}}{3\omega_{21}^2} + \tilde{x}. \quad (3.50)$$

The equation for  $\tilde{x}$  then becomes

$$\left(\frac{d\tilde{x}}{d\tilde{\sigma}}\right)^2 = 4\tilde{x}^3 - \tilde{g}_2\tilde{x} - \tilde{g}_3 \quad (3.51)$$

where

$$\tilde{\sigma} = \sqrt{\omega_{21}^2} \sigma \quad (3.52)$$

$$\tilde{g}_2 = \frac{4}{\omega_{21}^2} \left( v_{21}^2 + \frac{\omega_{21}^2 + \hat{A}}{3\omega_{21}^2} - \hat{A} \right), \quad (3.53)$$

$$\tilde{g}_3 = \frac{4}{\omega_{21}^2} \left[ \frac{(v_{21}^2 - \hat{A})(\omega_{21}^2 + \hat{A})}{3\omega_{21}^2} + \frac{2(\omega_{21}^2 + \hat{A})^3}{27\omega_{21}^4} - v_{21}^2 \right]. \quad (3.54)$$

Denoting the roots of the right hand side of (3.51) by  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$  we get

$$\left(\frac{d\tilde{x}}{d\tilde{\sigma}}\right)^2 = 4(\tilde{x} - \tilde{e}_1)(\tilde{x} - \tilde{e}_2)(\tilde{x} - \tilde{e}_3) \quad (3.55)$$

with the same relations as in (3.21), (3.22) but now  $\tilde{g}_i$  are given by (3.53) and (3.54). Using the transformation

$$\tilde{x} = \tilde{e}_1 + \tilde{e}_{21}\tilde{\eta}^2 \quad (3.56)$$

we get

$$\left(\frac{d\tilde{\eta}}{d(\sqrt{\tilde{e}_{31}}\tilde{\sigma})}\right)^2 = (1 - \tilde{\eta})(1 - \tilde{\kappa}\tilde{\eta}), \quad (3.57)$$

where

$$\tilde{\kappa} = \frac{\tilde{e}_{21}}{\tilde{e}_{31}}. \quad (3.58)$$

We note that the equation for  $\eta$  in spin chain case has exactly the same form as the equation (3.57) for  $\tilde{\eta}$  in the string case.

The solution of (3.57) is

$$\tilde{\eta} = sn(\sqrt{\omega_{21}^2 \tilde{e}_{31}} \sigma, \tilde{\kappa}). \quad (3.59)$$



and the solution for the original variable  $\sin^2 \psi$  is given by

$$\sin^2 \psi = \frac{\omega_{21}^2 + \hat{A}}{3\omega_{21}^2} + \tilde{e}_1 + \tilde{e}_{21} sn^2 \left( \sqrt{\omega_{21}^2 \tilde{e}_{31} \sigma}, \tilde{\kappa} \right). \quad (3.60)$$

The periodicity condition requires

$$\sqrt{\omega_{21}^2 \tilde{e}_{31}} = \frac{2K(\kappa)}{\pi}. \quad (3.61)$$

Now one can find the spins  $J_1$  and  $J_2$

$$J_1 = \omega_1 \left( 1 - \frac{\omega_{21}^2 + \hat{A}}{3\omega_{21}^2} - \tilde{e}_3 + \tilde{e}_{31} \frac{E(\kappa)}{K(\kappa)} \right) \quad (3.62)$$

and

$$J_2 = \omega_2 \left( \frac{\omega_{21}^2 + \hat{A}}{3\omega_{21}^2} + \tilde{e}_3 - \tilde{e}_{31} \frac{E(\kappa)}{K(\kappa)} \right). \quad (3.63)$$

As expected, the form of the spins  $J_1$  and  $J_2$  in spin chain and string cases are similar but not exactly the same. As we discussed in Section 2 the correspondence is valid only in one loop. To make the agreement exact, let us evaluate the terms which we have neglected in the action (2.22). First of all, we made change of variables as follows

$$\begin{aligned} t &= \kappa \tau; \quad \phi_i = \omega_i \tau + \alpha(\sigma), \\ \phi_1 &= \varphi_1 + \varphi_2 + t; \quad \phi_1 = \varphi_1 - \varphi_2 + t. \end{aligned} \quad (3.64)$$

The terms in the action which are small then become

$$\Delta S = \frac{R^2}{4\pi\alpha'} \int \left[ \dot{\psi}^2 + \dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2 \cos(2\psi) \dot{\varphi}_1 \dot{\varphi}_2 \right]. \quad (3.65)$$

Using the ansatz (3.64) we get

$$\begin{aligned} \dot{\varphi}_1 &= -\frac{(\kappa - \omega_1) + (\kappa - \omega_2)}{2} = -(\epsilon_1 + \epsilon_2), \\ \dot{\varphi}_1 &= -\frac{(\kappa - \omega_1) - (\kappa - \omega_2)}{2} = -(\epsilon_1 - \epsilon_2), \end{aligned} \quad (3.66)$$

where we set  $\epsilon_i = (\kappa - \omega_i)/2$ . The expression (3.65) then becomes

$$\begin{aligned}\Delta S &= \frac{R^2}{4\pi\alpha'} \int [\varepsilon_1 + \varepsilon_2 + 2\cos(2\psi)\varepsilon_1\varepsilon_2] \\ &= \frac{R^2}{4\pi\alpha'} \left[ \varepsilon_1^2\varepsilon_2^2 + 2\frac{\varepsilon_1\varepsilon_2}{\omega_1\omega_2} (J_2\omega_1 - J_1\omega_2) \right],\end{aligned}\quad (3.67)$$

where  $\varepsilon_1 = \epsilon_1 + \epsilon_2$ ,  $\varepsilon_2 = \epsilon_1 - \epsilon_2$ . Using the explicit expressions for the spins (3.62, 3.63) we find

$$\Delta S = \frac{R^2}{2\alpha'} \left[ (\varepsilon_1 + \varepsilon_2)^2 - 4\varepsilon_1\varepsilon_2(a - \tilde{e}_{31}\frac{E(\tilde{\kappa})}{K(\tilde{\kappa})}) \right], \quad (3.68)$$

where  $a = (\hat{\omega}_{21}^2 + \hat{A})/3\hat{\omega}_{21}^2 + \tilde{e}_3$ . From (3.67) one can conclude that  $\varepsilon_i$  are of order  $o(\alpha'/R^2)$ . Since

$$\begin{aligned}\omega_1 &= \kappa - (\varepsilon_1 + \varepsilon_2), \\ \omega_1 &= \kappa - (\varepsilon_1 - \varepsilon_2),\end{aligned}\quad (3.69)$$

the expression for  $\omega_1$  and  $\omega_2$  in one loop can be written as

$$\begin{aligned}\omega_1 &= \kappa - o\left(\frac{\alpha'}{R^2}\right), \\ \omega_1 &= \kappa - o\left(\frac{\alpha'}{R^2}\right).\end{aligned}\quad (3.70)$$

Redefining the integration constants so that the following equality holds

$$a = \frac{1}{2} - \left(\frac{B}{3\alpha} + \tilde{e}_3\right), \quad (3.71)$$

we find exact agreement between the solutions in spin chain model and rotating strings in one loop.

## 4 Conclusions

In this paper we considered a generalization of the solutions in spin chain/string duality proposed in hep-th/0311203 by Kruczenski. In section 2 we reviewed the proposed spin chain/string duality in the case of rotating string ansatz

elaborated in [17] leading to Neumann integrable system. In the next section we consider generalization to the solutions obtained in [16] using the ansatz (3.3)

$$\dot{\theta} = 0, \quad \phi = \omega\tau + \hat{\phi}(\sigma).$$

We found the solutions for the spin chain model and the corresponding energy and spins. Since the ansatz for the dynamical variables in this case are of more general form we suggested that the string solutions in this case should correspond to the more general string ansatz (3.35) leading to the Neumann-Rosochatius integrable system

$$\begin{aligned} \varphi_1 &= \omega_1\tau + \alpha_1(\sigma) \\ \varphi_2 &= \omega_2\tau + \alpha_2(\sigma). \end{aligned}$$

We calculated the conserved quantities in the case of two-spin solutions and found that the solutions are not exactly of the same form. To make the correspondence precise we consider in details the terms that are neglected when we calculated the string action in a certain limit used in [16]. The calculations show that, after appropriate redefinitions of the integration constants in the solutions, the expressions for spins and energy agrees exactly.

Although it is expected that the new solutions should be described in terms of spin chain model, it is interesting, and important, that we found exact agreement between the two models. It shows that this approach is valid for the most general known ansatz for rotating string and the result gives a strong support of the idea that there should be more direct way to establish the AdS/CFT correspondence of string theory in  $AdS_5 \times S^5$  background.

We note that it would be interesting to apply the same line of considerations to the Inozemtsev long range spin chain as discussed in [22].

*Note added:* After this paper was completed an interesting paper [23] appeared. In that paper it was proved that the results from spin chain and rotating strings agrees to the next order and a systematic procedure for computing the higher orders in large  $J$  expansion was suggested.

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